CLASSICAL, FUZZY, AND QUANTUM LOGICS: RELATIONS AND IMPLICATIONS

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ABSTRACT

In this paper, quantum logic is contrasted with both classical and fuzzy logics in order to highlight the fundamental differences implied by the different models of the world. Logic and set theory are treated in parallel in all the models, and the tools of both lattice theory and Hilbert space are used in the presentation. The elucidation of these models is significant for machine representation both of the external world and of that of human psychology. The conclusion summarizes differences between these variant logics. In a supplement we elaborate on quantum logic's nonclassical principle of operational ambiguity. An epilogue offers some speculations on the variant logics and human experience and suggests directions for future empirical investigations.

THE LATTICES OF CLASSICAL LOGIC

In order to reason about our experience, we use many mechanisms, but perhaps the best known and most studied is that of classical propositional logic and Cantor's set theory. These two methods are essentially the same if we allow the identification of a proposition with the set of all "possible worlds" in which the proposition is true. Specifically, classical logic posits a truth-value function which assigns a value of true or false to each proposition, where different truth-value functions correspond to different possible worlds (or models) for the propositions of interest.

As an example, there is an isomorphism between any physical system in classical mechanics and a collection of "experimental propositions" about its possible states, realized as a "field" of *subsets* of its phase space (Birkhoff and Von Neumann, 1936). Any such field of sets is isomorphic to a Boolean algebra (Stone, 1934), and thus to a classical logic. Conjunctions of propositions then correspond to intersections of the corresponding sets, disjunctions to unions, and negations to orthocomplements.

Each truth-value function corresponds, in turn, to a set-theoretic membership function (or "characteristic function") which assigns 1 to members of the corresponding set and 0 to nonmembers. This classical approach to reasoning is therefore a dichotomous one, often referred to as a "two-valued logic." In addition, the approach yields a "crisp set theory," because it answers a clear yes or no to the question of membership in a set of any given object. The conjecture that, in principle, there always exists an empirical procedure which can determine with certainty whether any object is or is not a member of some set is called the "postulate of fixed truth set" (Watanabe, 1969). For different reasons, we shall see that this postulate is rejected by both fuzzy and quantum logics.

The collection of all subsets of some universal set (or the collection of all propositions about some classical universe of discourse) has operations of intersection and union, making it a *lattice*: a set with two operations $\wedge$ (intersection or conjunction) and $\vee$ (union or disjunction) satisfying (for all x, y, and z in the set):

$$x \wedge x = x, \quad x \vee x = x \qquad \text{(idempotence)} \quad (1)$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad \text{(associativity)} \quad (2)$$

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x \qquad \text{(commutativity)} \quad (3)$$

The empty set (the false or absurd proposition) 0 and the universal set (the true or trivial proposition) 1 act as lattice identities: For all x,

$$0 \wedge x = 0, \quad 0 \vee x = x, \quad 1 \wedge x = x, \quad 1 \vee x = 1. \quad (4)$$

Adding a unique compatible complementation operator (write $x^{\sim}$ for the complement or negation of x), the lattice becomes an *orthocomplemented lattice*, or *ortholattice*, satisfying for all x and y:

$$(x^{\sim})^{\sim} = x \qquad \text{(involution)} \quad (5)$$

$$x \wedge x^{\sim} = 0, \quad x \vee x^{\sim} = 1 \qquad \text{(complement)} \quad (6)$$

$$(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \quad (x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}. \qquad \text{(DeMorgan)} \quad (7)$$

Finally, since the lattice satisfies the distributive property,

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{(distributivity)} \quad ,(8)$$

it forms an algebraic structure known as a *Boolean algebra.*

In classical logic, implication, $P \Rightarrow Q$ is equivalent to $Q \vee P^{\sim}$. To assert the "truth" of $P \Rightarrow Q$ is to say that $Q \vee P^{\sim} = 1$. It follows from distributivity that

$$P = P \wedge 1 = P \wedge (Q \vee P^{\sim}) = (P \wedge Q) \vee (P \wedge P^{\sim}) = (P \wedge Q) \vee 0 = P \wedge Q.$$

Therefore, $P \Rightarrow Q$ if and only if $P = P \wedge Q$. For sets, this same relation corresponds to "contained in":

$$x \subseteq y \quad \text{if and only if} \quad x = x \wedge y.$$

In general, containment imposes a partial ordering (reflexive, transitive, and antisymmetric) on the equivalence classes of propositions as lattice elements. A collection of objects having such a partial order relation is called a *partially ordered set*, or simply a *poset.* Thus we will simply write $x \leq y$ for $x = x \wedge y$ in a general lattice. However, one must be cautious when dealing with nonclassical lattices since the containment relation need no longer correspond to implication. Indeed, *material implication* ($P \Rightarrow Q$ iff $Q \vee P^{\sim} = 1$) cannot be defined on a quantum lattice (Fáy, 1967; Jauch and Piron, 1970; Oshins, 1989a).

Using this order relation, an additional property of the complement in a Boolean algebra is

$$\text{if} \quad x \leq y \quad \text{then} \quad y^{\sim} \leq x^{\sim} \qquad \text{(involutive anti-automorphism).} \qquad (9)$$

All Boolean algebras have two additional properties – *modularity* and *orthomodularity* (or *weak modularity*):

$$\text{if} \quad x \leq z \quad \text{then} \quad x \vee (y \wedge z) = (x \vee y) \wedge z \text{for all x, y, z} \qquad \text{(modularity),} \qquad (10)$$

$$\text{if} \quad x \leq y \quad \text{then} \quad y = x \vee (y \wedge x^{\sim}) \qquad \text{(weak modularity).} \qquad (11)$$

Some intuition can be gained for these properties by using lattice *Hasse diagrams* adapted from Oshins, Adelson, and McGoveran (1984). The basic partial ordering relation of the lattice is depicted in Figure 1. Hasse diagrams exhibit lattice partial orderings according to the convention that a bubble connected to a higher bubble with an upward directed line is contained within the higher bubble.

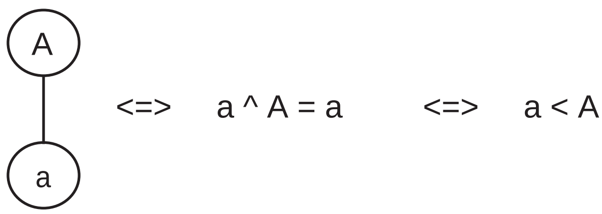

Figure 1. Hasse Diagram Exhibiting Partial Ordering.

Figure 2 shows the Boolean algebra associated with a Venn diagram of three nonintersecting sets and the Hasse diagram associated with three nonintersecting, Boolean points or atomic elements (atoms), where an atomic element P has the property: $0 < P$ and $x < P \Rightarrow x = 0$.

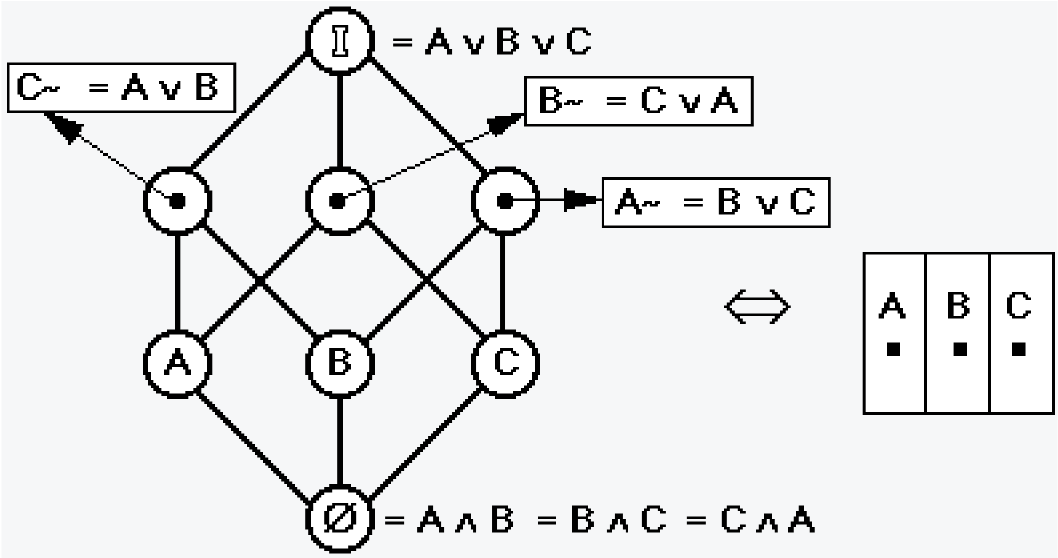

Figure 2. A Lattice of Classical Set Theory.

The result is a classical distributive lattice associated with three nonoverlapping unit elements which may be identified with equivalence classes of elementary atomic Boolean propositions or their associated Venn diagram. Negation is realized by reflection through the origin.

Figure 3 is a modular, nondistributive lattice representing subspaces of the plane. This nondistributive lattice is associated with three linearly dependent, coplanar, one-dimensional subspaces (lines) of the plane. The disjunction ("or") of two lattice propositions is their least upper bound and is realized as the span or plane generated by the associated lines. The conjunction ("and") of two lattice propositions is their greatest lower bound and is realized as the set-theoretic intersection of the associated lines, which determines a coordinate origin.

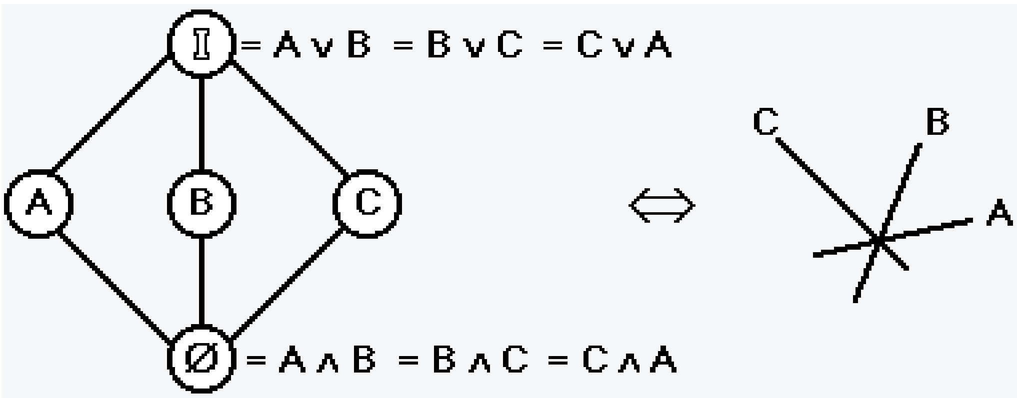

Figure 3. A Lattice of Subspaces of the Plane.

Furthermore, it is a simple exercise to show that distributivity implies modularity (but not conversely), and modularity and orthocomplemented imply orthomodular (but not conversely). Another helpful definition in this regard is that of *compatibility*: x and y are compatible, written $x \leftrightarrow y$ if and only if either

$$x = (x \wedge y) \vee (x \wedge y^{\sim}) \quad \text{or} \quad y = (x \wedge y) \vee (x^{\sim} \wedge y).$$

Using this definition, orthomodularity can be rephrased as saying

$$\text{if} \quad x \leq y \quad \text{then} \quad x \leftrightarrow y \qquad \text{(orthomodularity).} \qquad (12)$$

FUZZY LOGIC

An alternative approach to both logic and set theory arises from the fuzzy approach of Zadeh (1965), which starts with a truth-value function whose range is the interval from 0 to 1. This means that rather than being true or false, each proposition has a value which is supposed to provide a measure of its "likelihood." Rather than having worlds in which the proposition is true, we have worlds in which the proposition is more likely or less likely. For a given proposition P a fuzzy membership function μ_P is defined that maps each possible world into the number in the interval from 0 to 1 which is a measure of the likelihood of the proposition holding in this world. Thus the membership function is

$$\mu_P : \{\text{possible worlds}\} \longrightarrow (0,1).$$

(We consider here "strict" fuzzy logic, which never assigns the values 0 or 1; hence the range is the open interval (0, 1) excluding the endpoints, which provide a nonfuzzy limit of classical Boolean algebra or crisp set theory.)

In the fuzzy logic approach, the conjunction and disjunction of propositions are interpreted as in classical logic except that the truth values, expressed by the membership functions, are defined as

$$\mu_{P\wedge Q}(W) = min\{\mu_P(W), \mu_Q(W)\},$$

$$\mu_{P\vee Q}(W) = max\{\mu_P(W), \mu_Q(W)\}.$$

In other words, the likelihood of P and Q is the minimum of the likelihood of either one, while the likelihood of P or Q is the maximum likelihood of the two. It follows that $P \Rightarrow Q$ implies that $\mu_P \leq \mu_Q$ always.

In parallel with the correspondence between propositions and sets in a classical logic, *fuzzy sets* are defined via the fuzzy membership function, with fuzzy intersection and fuzzy union given by

$$(\mu_1 \wedge \mu_2)(W) = min\{\mu_1(W), \mu_2(W)\}.$$

$$(\mu_1 \vee \mu_2)(W) = max\{\mu_1(W), \mu_2(W)\}.$$

There is now once again a one-to-one correspondence between the equivalence classes of propositions of fuzzy logic and fuzzy sets (although in the "strict" fuzzy logic the propositions true and false are discarded), and in either case a lattice is obtained satisfying the lattice properties (1) through (3) above. In the strict fuzzy theory, however, the lattice has no identities 0 and 1. It is therefore impossible for the lattice to have ordinary orthocomplements or to qualify as a Boolean algebra. Nevertheless, in fuzzy theory there is a *fuzzy complement* $P^\wedge$, defined for each proposition P as the proposition satisfying

$$\mu_{P^\wedge}(W) = 1 - \mu_P(W).$$

Correspondingly, each fuzzy set has a complement given as

$$\mu^{\wedge}(W) = 1 - \mu(W).$$

Notice that this complement satisfies involution (5) and DeMorgan's Laws (7), but complementation (6) becomes the weak statement

$$0 < \mu \wedge \mu^{\wedge} = min\{\mu, 1-\mu\}, \quad \mu \vee \mu^{\wedge} = max\{\mu, 1-\mu\} < 1. \tag{6'}$$

If instead of strict fuzzy logic, true and false were included, with $\mu_{true} = 1$ and $\mu_{false} = 0$, the identities rule (4) would also hold. However, although the lattice would now contain its limit points, the fuzzy complement would still not be a lattice complement since, in general, the complement rule (6) would still not hold. In addition, for a fixed proposition in fuzzy logic the partial ordering becomes a *total* ordering relationship on the set of worlds, since any two fuzzy memberships, and their connectives as maximums and minimums, are comparable. Finally, the distributive rule (8) still holds for fuzzy sets, and as a result, modularity holds. Of course, without a true complement in a strictly fuzzy lattice, orthomodularity is out of the question.

QUANTUM LOGIC

The development of modern quantum physics has revealed that there are natural phenomena which cannot be described and reasoned about in the framework of classical logic. The behavior of elementary particles exhibits a property that can be phrased in the form "P or Q is true even though P is not true and Q is not true." For example, "light is a wave or particle phenomenon" even though neither one describes light adequately. As Bohr wrote in 1929 (1961, p. 10): quantum experience,

> forces us to adopt a new mode of description designated as complementary in the sense that any given application of classical concepts precludes the simultaneous use of other classical concepts which in a different connection are equally necessary for the elucidation of the phenomena (1961, p.10).

As such, complementary constructs provide competing alternative frames of reference: WAVE or PARTICLE = 1 (necessary), and WAVE and PARTICLE = 0 (mutually exclusive), but WAVE $\neq$ 1 and PARTICLE $\neq$ 1. As mentioned below, unlike the classical case, "mutually exclusive" does *not* imply orthocomplemented (i.e., $A \wedge B = 0 \;\; A \subseteq B^{\sim}$).

More specifically, well-known experiments (Dirac, 1958; Feynman, 1951; Feynman and Hibbs, 1965; Feynman, Leighton, and Sands, 1965; Noyes, 1980) show that when light is passed through two slits in an opaque screen an interference pattern is formed on a photosensitive plate on the other side of the screen, even when only a *single photon* (irreducible packet of electromagnetic energy) is projected at a time. Expressed succinctly by Dirac:

> It would be quite wrong to picture the photon and its associated wave as interacting in the way in which particles and waves can interact in classical mechanics. [In order to avoid] contradict[ing] the conservation of energy ... [e]ach photon ... interferes only with itself. Interference between two different photons never occurs (1958, p.9).

The photon must pass through one slit or the other slit, but the observed behavior is not that of a photon which has simply passed through a single slit. As will be discussed below, this can be experimentally verified by covering one of the slits, at which point the interference behavior disappears. Similar arguments apply for other individual quanta, such as electrons, which are classically thought of as particles and not as waves (Feynman, 1951). Although double-slit interference provides for determination of continuous degrees of freedom (i.e., measurement of position/location), similar arguments can be presented for discrete alternatives such as for the quantum dichotomy spin in a Stern-Gerlach apparatus (Oshins, 1989b; Oshins and McGoveran, 1980).

In quantum formalisms, one considers a type of encoding of patterns and of regularities in physical experience that is at fundamental variance with classical parallel-process modeling, such as either the generic McCulloch-Pitts-type neural net or Pribram's "hologram hypothesis" (Hilgard, 1989; Oshins, 1984, 1989a). Quantum parallel processing occurs when two empirical processes are possible but there is no empirical procedure that is capable of discriminating which one has actually occurred. It is then *empirically* meaningless to speak about there being individual events (in the classical sense of having a fully defined and fixed sample space). In our supplement we provide a technical summary of how quantum logic offers a type of "operational ambiguity" as a substitute for the distributive law of classical and fuzzy logics.

Returning to our example of the double-slit interference experiment, let us examine further how to provide a logic in which to realize these quantum phenomena. In the aforementioned experiment with the photons and slits (see Figure 4), if a photon is known to pass through slit A (by covering B, for example), it arrives at the plate according to a probability distribution centered at A'; if, on the other hand, it is known to pass through slit B, it must arrive according to a distribution centered at the point B' on the plate.

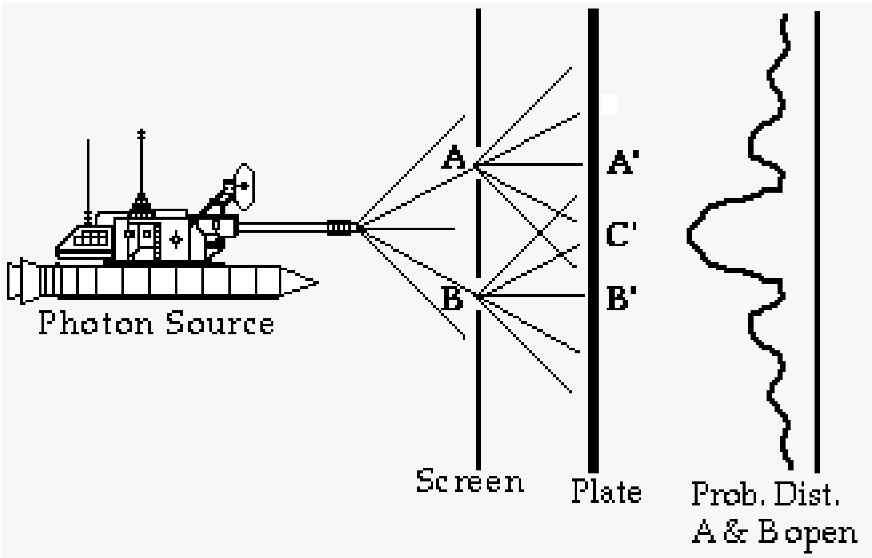

Figure 4. An Experiment Splitting a Light Beam.

Consider the following two cases: case I in which both slits A and B are open and case II in which only one slit is open. The following facts are true: (1) Where a minimum occurs in the case I distribution of quanta (photons or electrons), case II would increase the number found; and (2) at the center of the case I distribution, where the case I maximum occurs at C', case II would substantially decrease the number found. Thus, it is as though closing one hole has both increased and decreased the number of quanta which pass through the other hole, which appears to be illogical.

From a probabilistic/likelihood perspective, letting Q be the statement "the photon passed through slit A," R be the statement "the photon passed through slit B," and P be the statement "the photon arrived near point C' on the plate, we conclude that $P \wedge Q$ is false in almost all cases and $P \wedge R$ is also false in almost all cases. Consequently, $(P \wedge Q) \vee (P \wedge R)$ is false in almost all cases. Nonetheless, when a photon is projected, it is most frequently observed to arrive near point C'. Since the photon is indivisible, it must have come through one of the slits or the other; therefore, whenever P is true, $P \wedge (Q \vee R)$ is true. This contradicts the distributive law (8)!

Although the foregoing example raises the issue of probability in regard to the behavior of elementary particles (see also below) and expresses results in terms of distributions, it must be emphasized immediately that, unlike the fuzzy logic approach, there is no uncertainty or limit to precision implied regarding the ability to measure a single degree of freedom for an individual quanta at one time – it is possible to measure with great precision the point of arrival of a single photon on the plate of Figure 4. Therefore, an individual photon either does or does not arrive at a point near C', and, accordingly, $P \wedge (Q \vee R)$ is either completely true or completely false for the given photon. The set of photons satisfying $P \wedge (Q \vee R)$ is therefore a crisp set.

The allegation that $P \wedge (Q \vee R)$ is not equal to $(P \wedge Q) \vee (P \wedge R)$ is to be interpreted empirically. In experiments in which it is possible to measure $P \wedge Q$ or $P \wedge R$ (by covering one of the slits) the results obtained differ greatly from those in which $P \wedge (Q \vee R)$ is measured (by leaving both slits open). The probability distributions for photons passing through individual slits, separately, do not add up to the distribution for individual, indivisible photons that are allowed to pass through both slits. Since the photon passes through (one slit or the other) being true does not allow us to conclude empirically that the photon passes through one slit is true nor that the photon passes through the other slit is true, we can say that for photons under appropriate conditions the truth value of "to pass through" does not distribute over the complete set of alternatives understood as the collection of slits through which it must pass. [The reader is cautioned that the language used by Feynman (1951, p. 536) and Feynman and Hibbs (1965, p. 13) incorrectly distributes "to pass through."]

One widely used construction of a quantum logic (Birkhoff and Von Neumann, 1936; Finkelstein, 1963; Greechie and Gudder, 1974; Jauch, 1968, 1972; Jauch and Piron, 1969, 1970; Mackey, 1963; Oshins and McGoveran, 1980; Piron, 1976; Von Neumann, 1955) is based on a collection of propositions (or "questions" or "experiments") about a physical system which are related by $x \leq y$ if and only if for every state of the system in which x is always found to hold, y is also always found to hold. (One must actually be a little more careful by defining the elements of the logic to be equivalence classes of propositions so as to ensure that $x \leq y$ and $y \leq x$ implies $x = y$. We will assume this has been done and ignore the formal details.)

An empirical partial ordering of containment is obtained on propositions with truth-valuation given by empirical tests. Hence, a conjunction, $x \wedge y$, can be operationally defined in terms of this empirical partial ordering of containment of empirical tests as the experiment z such that (Finkelstein, 1963):

$x \wedge y = z$ formally means:

- $z \leq x, \quad z \leq y$
- If $w \leq x$ and $w \leq y$, then $w \leq z$

$x \wedge y = z$ operationally means:

- All preparations which "pass" the test of $x \wedge y = z$ will pass a determination of empirical test-x and will pass a determination of empirical test-y.
- All preparations which pass both the test of x and the test of y will pass a determination of the test of $x \wedge y = z$.

That $x \wedge y$ is the greatest lower bound of x and y can be seen since: (1) if $z \leq x$ and $z \leq y$, then in any situation that z is necessarily true, both x and y must be true; consequently $x \wedge y$ must be found to be true also. Conversely, (2) for $x \wedge y$ always to hold in some state of the system it must be that x always holds and y always holds.

Although this definition provides an experimental meaning to a propositional conjunction $x \wedge y = z$ in terms of propositions about component events x and y if it exists, it does *not* show how to test for $x \wedge y$ in terms of the component events x and y. We do not have a guarantee that there exists such a test. The reason for this is that quantum theory requires the existence of incompatible events such that the determination of x followed by the determination of y does not yield the same empirical outcome as the determination of y followed by the determination of x.

We can understand this difficulty in defining the conjunctive proposition as follows: Identify the component propositions x and y with their projection operators Mx and My (defined in the next section), say, interpreted as filters (Oshins, 1989b). The difficulty in defining the conjunctive proposition occurs since for incompatible propositions which can not be simultaneously realized, the projection operators do not commute, $M_x M_y \neq M_y M_x$. Thus, the ordering of their experimental tests for the component proposition is critical.

Jauch (1968, p. 75) has suggested a way to deal with this by identifying the conjunctive proposition $x \wedge y = z$ with an infinite sequence of alternating pairs of filters for the propositions x and y respectively:

$$M_{x\wedge y} = lim_{n\to\infty}(M_x M_y)^n$$

The proposition $x \wedge y = z$ is then true if the system passes the experimental test corresponding to this filter, and false otherwise. Clearly, this definition is consistent with the classical and fuzzy definitions for compatible propositions. One consequence of the quantum conjunction is that for incompatible propositions x and y, $z = x \wedge y = 0$, does not imply that $x \leq y^{\sim}$, nor that $y \leq x^{\sim}$, as it does in the classical case.

Similarly, a disjunction $x \vee y = z$ can be operationally defined in terms of the empirical partial ordering of containment of empirical tests as the experiment z such that (Finkelstein, 1963):

$x \vee y = z$ formally means:

- $x \leq z, \quad y \leq z$
- if $x \leq w$ and $y \leq w$, then $z \leq w$

$x \vee y = z$ operationally means:

- All preparations which pass the test of x will pass a determination Of $x \vee y = z$, and likewise all preparations which pass the test of y will pass a determination of $x \vee y = z$.
- Any determination which passes a preparation for x and also passes a preparation for y will pass a preparation for $x \vee y = z$.

It is easy to show that $x \vee y$ is the least upper bound of all propositions z which are $\geq x$ and $\geq y$. Finally, an orthocomplement can also be defined: $x^{\sim}$ asserts to perform the same experiment as for x but answer true if and only if x yields false. The resulting structure is an orthomodular lattice of propositions [properties (1) - (7) and (11)].

A collection of (σ-additive, positive) functionals, called *states*, from the lattice to the interval [0, 1] are now added to represent the set of all true propositions of the system. As such, a physical state is considered to result from an empirically meaningful "preparation" of the system. A state is then "determined" by measuring the truth or falsehood of a "maximal" set of compatible propositions of the system.

Classical states have the property that every proposition necessarily is either true or is false. In distinction, strictly fuzzy states have the property that every state is both true and false, somewhat. In contrast, in quantum logic although a predicate will have a crisp value upon being determined, not every proposition is necessarily true nor necessarily false. Jauch and Piron caution that although one predicts the probabilities of outcomes for various measurements, given specified experimental arrangements, quantum states and quantum physics pertain to individual quantum systems and are not intrinsically probabilistic as is sometimes thought: "the state is a property of an individual system and not of a statistical ensemble of such systems. ... a probability is meaningful only with reference to a statistical ensemble. The definition we have given above refers only to true propositions, that is to what we have called properties of the system, and there is no objection in attributing these properties to an individual system" (1969, p. 846).

Each state, s, satisfies the properties:

- $s(0) = 0, \quad s(1) = 1,$
- $s(x \vee y) = s(x) + s(y)$ for all x, y with $x \leq y^{\sim}$,
- *separating alternatives*: If $x \neq 0$, then there exists a state s such that $s(x) \neq 0$; and if $x \neq y$, then there exists a state s such that $s(x) \neq s(y)$ (Jauch, 1968, pp. 94-96).

Moreover, the collection of states must be "full," meaning that they can be used to determine the order on the lattice:

$$\text{for all } x, y: \quad x \leq y \quad \text{if and only if} \quad s(x) \leq s(y) \qquad \text{for all states s.}$$

This corresponds to the idea that if x is not $\leq y$ then there should exist some state of the system referred to by these propositions in which it is more likely that x is true than that y is true. That quantum states, as (σ-additive, positive) functionals from the lattice to [0, 1], are essentially unique, has been shown by Gleason (1957).

States can be thought of as representing possible worlds, as do the truth functions of classical and fuzzy. They can therefore be used to define something akin to a fuzzy membership function: for any given x in the lattice, define

$$\mu'_x(s) = s(x) \quad \text{for all states s.}$$

A significant difference between the quantum μ'_x and the fuzzy μ_p is in the conflicting requirements whenever $x \leq y^{\sim}$:

$$\mu_{P \vee Q}(W) = max\{\mu_p(W), \mu_Q(W)\},$$
$$\mu'_{x \vee y}(s) = s(x \vee y) = s(x) + s(y) = \mu'_x(s) + \mu'_y(s).$$

Although fuzzy and quantum logics appear to share a likelihood approach to the problem of describing possible worlds, this similarity is superficial. The fuzzy logic approach treats the ideas of truth and membership themselves as not binary, whereas in quantum logic they remain binary as in classical logic. Probabilities in quantum logic refer to the expectation value of ensemble outcomes of repeated experiments which, on any given execution, return a clear-cut yes or no. Where a fuzzy proposition such as "Joe is tall" has a value of, say, 0.8 in a world in which Joe is six feet tall, classical/quantum propositions such as "the toothbrush/electron is found in the left compartment of the suitcase/box" has a value of 1 in a world in which, upon performing an empirical test such as opening the suitcase/box and looking, the toothbrush/electron is found in the left compartment.

In a world in which the owner of the suitcase tends to keep his or her toothbrush in the left side, the corresponding state for a classical or quantum logic might assign s(toothbrush in left side) a value of 0.8, but this is not to imply that when you open the suitcase you will not know with certainty that the toothbrush is or is not in the left-hand compartment, only that you can expect to find it there 80% of the time when you look.

Quantum logic is distinguished dramatically from classical and fuzzy logics due to the underlying nondistributivity of the quantum lattice of propositions. A nondistributive orthomodular lattice of propositions divides itself into subsets of compatible (see "$\leftrightarrow$" above) propositions. Incompatible propositions are those which represent measurements or experiments that cannot be performed simultaneously. For example, the Heisenberg uncertainty principle (Schiff, 1968, pp. 7-14) asserts that measurement of the momentum of an elementary particle and measurement of the position of the same particle are incompatible in this sense: empirically, there is no state of simultaneous position and momentum. This and the photon experiment presented earlier cannot be described in terms of fuzzy logic but do fit the quantum logic scheme.

One of the difficulties of working with quantum logics is that the structure of orthomodular lattices is complex and not well understood. Despite a remarkable representation theorem of Piron (1976), in contrast to the Boolean algebras of classical logic, which are very regular and always representable as sets of subsets, orthomodular lattices can boast no such simple characterization (Anger, Sarmiento, and Rodriguez, 1986; Kalmbach, 1983). On the contrary, researchers have been able to show extreme examples such as orthomodular lattices on which it is impossible to define even one state (Greechie, 1971). As noted by Jauch (1968, p. 96), there are also Boolean lattices that have no states at all.

The next section presents the Hilbert space projection lattice, which has served in place of the "sets of subsets" representation of classical logic; nevertheless, Greechie (1969) has shown that there are orthomodular posets that cannot be represented even in this manner. Despite these difficulties, the need for non-classical models is evident in describing quantum phenomena, and there is also considerable evidence that quantum logic might provide a better model for many human thought processes than do classical or fuzzy models (Hilgard, 1989; Oshins, 1984, 1989a,b; Oshins and McGoveran, 1980; Shepard, 1989, pp. 123-125). Our Epilogue and Supplement briefly suggest psychologically related phenomena which might have quantal or quantum-like realizations.

LOGIC AND HILBERT SPACE

Hilbert space has long been used in quantum mechanics as the natural environment for expressing quantum phenomena (Von Neumann, 1955; Mackey, 1963). Von Neumann, Birkhoff, Mackey, Jauch, Finkelstein, and Piron laid much of the mathematical foundations. Bohr, Von Weizsacker (1955, 1958, 1970, 1971), Heisenberg, Finkelstein, and Watanabe have played key roles in developing the interpretive structure.

The lattice of closed subspaces in Hilbert space provides a metalogic for our logic (Heisenberg, 1958; Oshins, 1989a,b; Oshins and McGoveran, 1980; Von Weizsacker, 1955, 1958), since it contains lattices of subspaces which are orthomodular and nondistributive. Closed subspaces, as rays, correspond to "projection operators" on a projective Hilbert space, modulo the complex numbers, and represent orthogonal projections of the space on one-dimensional subspaces. Since such linear operators can be represented by (possibly infinite) matrices, the quantum logic can be represented as a class of matrices over the complex numbers having the properties:

$$M^2 = M \qquad \text{(idempotent)} \qquad (M1)$$

$$M \text{ commutes with its adjoint (or conjugate transpose):} \quad MM^* = M^*M \qquad \text{(normal)} \qquad (M2)$$

Hilbert spaces are complete inner-product (vector) spaces and hence have a concept of orthogonality: two vectors are orthogonal if their inner product is zero. Similarly, two projection operators are "orthogonal" if their associated subspaces (images) are orthogonal. The orthogonal complement, $M^{\perp}$, of a projection matrix M is another projection matrix such that

$$M^{\perp}M = MM^{\perp} = 0 \quad \text{(the zero matrix)}.$$

Finally, the intersection of closed subspaces V and W is a closed subspace $V \wedge W$ while $V \vee W$ is defined as the smallest closed subspace containing both V and W (the *span* of V and W). It is a fundamental theorem of the field that the closed subspaces of a Hilbert space form an orthomodular but non-distributive lattice, called the *projection lattice* of the space (Beran, 1984; Piron, 1976; Von Neumann, 1955).

In order to represent fuzzy logic in terms of projection operators on Hilbert space (Oshins, Adelson, and McGoveran, 1984), we consider a particular lattice of propositions L and fix a proposition P of L. Since μ_P maps possible worlds into the interval (0, 1), it induces a total order on the set of worlds:

$$W_1 \leq W_2 \quad \text{if and only if} \quad \mu_P(W_1) \leq \mu_P(W_2).$$

If M_P is a matrix whose rows are put into one-to-one, order-preserving correspondence f, with a sequence of points from 0 to 1, we can think of placing along the diagonal the values M[k, k] = 1 if $f(k)$ is in the range of μ_P, and M[k, k] $= 0$ otherwise. This makes the fuzzy membership matrix M_P into a normalizable direct sum of orthonormal projection operators. Thus we satisfy the above conditions by means of compatible projection operators.

In particular, for any such matrices, M and N,

$$MN = NM \text{ (commutative)}, \quad M^* = M\text{(self-adjoint)},$$

where the "spectrum," or set of eigenvalues of the component projection operators, is just $\{0, 1\}$. However, the orthogonal complement $M^{\perp}$ does not correspond to the fuzzy complement, $\mu^{\wedge}$. In fact,

$$M^{\perp}[k,k] = 1 - M[k,k] \quad \text{for all k},$$

or

$$M^{\perp}[k,k] = 1 \quad \text{if and only if} \quad M[k,k] = 0,$$

compared with

$$M^{\wedge}[k,k] = 1 \quad \text{if and only if} \quad f(k) \text{ is in the range of } \mu_{P^{\wedge}} = 1 - \mu_P$$

[i.e., if and only if 1 - $f(k)$ is in the range of μ_P].

If one defines a legitimate orthocomplement on the lattice corresponding to completed fuzzy matrices, there exists a unitary transformation that transforms between the orthocomplement and the fuzzy complement. By adjoining all the orthogonal complements, an orthocomplemented projection lattice is obtained for fuzzy logic. Since this lattice is also distributive, it is therefore a Boolean algebra, just as in the case of classical logic.

CONCLUSION

By contrasting the lattice properties of classical, fuzzy, and quantum logics, it is possible to make a comparison of the power or weakness of the three logics. Classical logic produces a Boolean algebra, fuzzy logic produces a distributive but not complemented lattice, and quantum logic produces an nondistributive orthomodular lattice. By representing membership functions by projection operators on Hilbert space, it is possible to add orthocomplements to the fuzzy lattice in a natural way and thereby obtain a Boolean algebra. In terms of the lattices, it is seen that the fuzzy logic lattice is somewhat more general than the classical by not requiring orthocomplements, but it is less general than quantum logic by requiring distributivity. Fuzzy allows discussion of worlds which do not have orthocomplements, but since fuzzy disallows orthocomplementation, it prevents the discussion of incompatible pairs of worlds. The fuzzy approach addresses the fuzzy set of people who Bill likes with no concern for dividing the world into liked and unliked people. The quantum approach considers the set of circumstances in which photons/electrons always have position or momentum if looked for, but do not have position and do not have momentum independent of the experimental determination as a metacontext. All the photons that reached the plane behind the screen went through slit A or slit B, but did not just go through slit A nor just through slit B. The other alternative is relevant.

EPILOGUE

A final illustration: a suitcase is constructed with two compartments in such a way that only one compartment can be opened at a time. A particular shirt may be in the left side or right side. Classical logic would divide the set S of possible worlds in which the shirt is in the suitcase into two mutually exclusive and all inclusive subsets: A = worlds in which the shirt is in the left side, and B = worlds in which the shirt is in the right side. Thus,

$$S = A \vee B \quad \text{and} \quad 0 = A \wedge B.$$

Fuzzy logic would ascribe to each of the possible worlds a likelihood that the shirt is in the left side and a likelihood that it is in the right side and thereby create a likelihood equal to the minimum of these two that the shirt is in both the left and the right sides. Furthermore, the likelihood of the shirt being in the suitcase (i.e., in the left or right sides) will be assigned the maximum likelihood of its being in the left or in the right sides. This yields the peculiarity that although the shirt is always in the suitcase with certainty, its fuzzy likelihood of being in the left or right sides is less than 1!

Lastly, quantum logic would consider only worlds in which a "quantal shirt" would be *found* to be in the right side or *found* to be in the left side. If P is the proposition "found on left" and Q = "found on right," then any state s will satisfy

$$s(P) + s(Q) = s(P \vee Q) \quad \text{(because } Q \leq P^{\sim}\text{)}.$$

The set of worlds for which the shirt is always found in the right side is the set of states for which s(Q) = 1. There may also be states for which, say, s(Q) = 0.7 and s(P) = 0.3, or for which s(Q) = 0.5 and s(P) = 0.5, but here $s(P \wedge Q) = s(0) = 0$ and $s(P \vee Q) = 1$. There is no possibility that the shirt is in both sides or nowhere at all.

This strange suitcase is modeled on the original Birkhoff and Von Neumann example of complementary alternatives in a nondistributive lattice [see also Heisenberg (1958) for Von Weizsacker's interpretation as metalogic]:

> facts suggest that the distributive law *may* break down in quantum mechanics. That it *does* break down is shown by the fact that if a denotes the experimental observation of a wave-packet y on one side of a plane in ordinary space, $a^{'}$ correspondingly the observation of y on the other side, and b the observation of y in a state symmetric about the plane, then ... [the distributive law is violated]. (1936, p. 83)

Here we see that if an individual quanta is looked for, it will always be found in its entirety on either the left or the right side of the plane. Thus, (left or right) will always be true, if looked for. Yet, since cosines, which are symmetric, and sines, which are antisymmetric, provide a complete basis for locating the quanta, we can assert that (even or odd) will always be true, if looked for. Yet, (even and left) is false and, likewise, (even and right) is false. Therefore,

$$even \wedge (left \vee right) \neq (even \wedge left) \vee (even \wedge right),$$

and distributivity fails.

There are other effects – both quantal and psychological – which seem to exhibit behavior inconsistent with classical or fuzzy logic. The "liar's paradox" realized as "This statement is false" has been suggested by Oshins (1989b; Oshins and McGoveran, 1980) as a linguistic example of a failure of the distributive law. Specifically, Oshins has reinterpreted the "liar's paradox" as "This statement is true or false" does not imply that "This statement is true" nor that "This statement is false," which is equivalent to a rejection of the distributive law. Instead, the truth valuation induces a transition in truth-value within an irreducible proposition.

Another result related to the failure of distributivity is the Aharonov-Susskind effect (Aharonov and Susskind, 1967; Bernstein and Phillips, 1981), which has demonstrated empirical consequences to relative rotations by 2π of a quanta with respect to its environment. If a single electron is contained in a box having reflecting walls and two compartments that are separated by an impenetrable partition and if the electron is initially found in the left compartment, it follows that the electron's state can neither be even nor odd, and indeed the individual electron's state must be a superposition of even and of odd states. Thus, if the partition is initially open, so that the electron can freely go back and forth, one can show that there exists a probability current for this to occur due to the superposition of even and odd states. Furthermore, if one closes the partition so that it is temporarily impenetrable, applies a uniform magnetic field, and then rotates one compartment 360^o with respect to the other compartment, one can show that upon removing the impenetrable partition, the probability current has reversed direction.

To see how this might pertain to psychophysical measurements consider (Bernstein and Pillips, 1981; Oshins, 1984): If one picks up an object, such as an alarm clock, and rotates it 360^o with respect to some reference frame, such as oneself; it ends up where it started. In contrast, if one holds one's hand with the palm up, as if holding a cup of tea on the palm, and rotates it, while keeping the palm up so as not to spill the tea, one finds it necessary to turn the palm around the rest of oneself *twice* (i.e., 720^o, once below the elbow and once above it), before one winds-up where one is/was! Oshins has described this property as "self-referential motion."

Noting that (1) in mentally comparing differentially oriented, asymmetrical geometric objects, the time required to accurately discriminate whether or not a second object is the mirror image or an identical object is linearly proportional to the relative angular orientation – thereby indicating that rotations of mental representations take place, cognitively, in these comparisons (Shepard and Metzler, 1971); and that (2) by applying an alternating direct current to a finger and using a SQUID (superconducting quantum interference device), one obtains a highly localized magnetic activity site (dipole moment) on the opposite side of the skull (Williamson, Kaufman, and Brenner, 1979), Oshins (1984) has suggested that SQUIDs might be able to measure *quantal* magnetic activity in the brain as a result of self-referential motion – a psychological

Aharonov-Susskind effect.

Another possible candidate for a quantal perceptual effect might be Necker's cube. The bistable perceptual effect encountered with Necker's cube might be considered a quantal effect in which the state is in configuration-1 or configuration-2, but is not in configuration-1 and is not in configuration-2 – thus violating the distributive law.

Thus, we see that the assumption of a quantum logic representation for psychology leads to alternative conceptions of formal linguistic and psychophysical processes. Yet, since quantum logic contends to be an empirical logic, whether or not there is a substantive role for quantum logic in psychology awaits experimental confirmation and, ultimately, the determination of nature.

SUPPLEMENT ON OPERATIONAL METALOGICAL AMBIGUITY

As an alternative to the distributive law of classical and fuzzy logics, quantum theory provides a type of fundamental, "operational ambiguity" (Oshins, 1989a). This is done by means of symmetries on irreducible equivalence classes of alternative, competing possibilities, realized as a (nondistributive, orthomodular, atomic) lattice (Oshins, Adelson, and McGoveran, 1984). The symmetries of the observable structure are generated by the observables themselves. One-dimensional projection operators (see "Logic and Hilbert Space," above) can be used to construct the observables through their spectral decompositions into projection-valued measures. They can be identified with the atoms of the lattice and with rays in the projective, carrier representation space. This can also be done for classical and fuzzy logics.

Since classical and fuzzy logics do not admit incompatible propositions, as a result of being distributive, their spectral realizations are constructable from commuting projection operators. As a result, the products of their projection operator give delta functions (Oshins et. al., 1984), in contrast to the quantum case where the products give (hyper-)complex numbers (Oshins et. al., 1984; Feynman, 1951; Schwinger, 1970).

Wigner's theorem (Wigner, 1959) asserts that in order to preserve the probability structure of the state space, which carries all the information about possible events, one requires symmetries (e.g., change) to be projective and unitary or antiunitary. The unitary operators are constructable from the observables through their parameterized exponential mapping. Under somewhat weaker requirements, the generalized Wigner-Uhlhorn theorem shows that this symmetry structure can then be identified with the *fundamental theorem of projective geometry* (Baer, 1952; Biedenharn and Louck, 1981, pp. 157-201); Uhlhorn, 1962).

Although clearly such more general quantum representations cannot be formally accomplished within classical models, there do exist limits and contractions (e.g., Dirac's contraction from quantum commutator brackets into classical Poisson brackets, which exhibits the differential structure of classical physics as a limit of the differential structure of quantum physics) that enable one to construct approximations to classicalness given certain circumstances. Another example is Ehrenfest's theorem (Schiff, 1968, pp. 28-30) which exhibits the equations of motion of classical observables as ensemble expectation values of the corresponding quantum observables.

The quantum process is described as a "non-discriminating measurement" (Schwinger, 1970) of "interfering alternatives" (Feynman and Hibbs, 1965) realized by an irreducible or "coherent" (Jauch, 1968) lattice of propositions. The equivocation process provides a type of irreducible, representational *ambiguity* that does not exist in classical representations, such as in computers (Deutsch, 1985; Feynman, 1982; Gornitz and Von Weizsacker, 1987; Oshins, 1984, 1989a,b; Oshins and McGoveran, 1980). Instead of trying to avoid ambiguity, Oshins has elevated this specific type of ambiguity into a fundamental principle (Hilgard, 1989; Jauch, 1968, p. 106; Oshins, 1984, 1989a,b):

> If one cannot (operationally) distinguish [or discriminate] between two unit predicates A and B, there will always exist a third possible contrary (unit) predicate C such that (A or B) = (B or C) = (C or A), i.e., they are equivalent perspectives – there is no operational way to distinguish [or discriminate]

between A, B, and C. (Oshins 1989a, p. 5)

Operationally, quantum physics has shown that there is an empirical difference in coding between (Oshins, 1984, 1989a,b; Schwinger, 1970, esp. pp. 27-28): (1) forming a class (virtual ensemble) of possible states (subensembles) which *are distinguished* by some specific attribute (e.g., subensembles having either predicate-A or incompatible/complementary predicate-B) and (2) forming a class (virtual ensemble) of possible states (subensembles) which are *not empirically distinguishable* according to such a predicate (i.e., there is no empirical procedure that discriminates between the alternative, possibly incompatible/complementary predicates). This is true *even if no member of the class is distinguished or separated out*, as long as, operationally, one could be.

Oshins (1989b) has suggested a possible psychological example which might carry this same type of representational alternatives. Consider the difference between saying a person (which would always equal male or female, if gender were to be distinguished) came into the room and saying that a male or a female came into the room (actually distinguishing gender as opposed to a different, competing context). Of course, the existence of complementary, competing construct attributes is an empirical issue. Other possible complementary alternatives might involve a metachoice between, say, the good/bad attribute dicotomy and the love/hate attribute dicotomy. See also Bohr (1987/1954, p. 81) regarding "justice and charity" and Heisenberg (1958, p. 179) regarding "enjoying music and analyzing its structure." The existence of two operational ways to code "non-selecting measurements" became a foundation for Oshins' "quantum psychology" approach (Oshins, 1984, 1989a,b).

ACKNOWLEDGEMENTS

We are indebted to Lou Kauffman, David McGoveran, and H. Pierre Noyes for their helpful comments on this paper. We particularly thank Jeff Yerkes for the appearance of this chapter and for coordinating the sometimes extensive communications between the authors.

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